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# CONTINUOUS FUNCTIONS, REVISITED

**Prop:** For metric spaces $(X, d_X)$ and $(Y, d_Y)$, the $\varepsilon$-$\delta$ definition of a continuous function and the topological definition coincide.

**Proof:**

## Standard Examples

Let $X, Y$ and $Z$ be topological spaces.

1. **Constant function:** Let $f: X \longrightarrow Y$ be a function s.t. for some $y_0 \in Y$, $f(x) = y_0 \ \forall x \in X$.
   Then $f$ is continuous, $\because$ the inverse image of any set is either $\emptyset$ or $X$.

2. **Restricting Domain:** If $f: X \to Y$ is continuous and $A \subseteq X$, then in the subspace topology, the restricted function $f|_A : A \to Y$ is continuous.

   **Proof:** Let $g = f|_A$.
   $\forall V \subseteq Y$ open,
   $$g^{-1}(V) = f^{-1}(V) \cap A, \text{ which is open in } A.$$

3. **Restricting / Expanding Range:** Let $f: X \longrightarrow Y$ be continuous. (i) Given a subset $Z$ of $Y$ s.t. $f(X) \subseteq Z$, the function $g: X \longrightarrow Z$ obtained by restricting the range of $f$ is continuous.
   (ii) If $Y \subseteq Z$, then $h: X \longrightarrow Z$ obtained by expanding the range of $f$ is continuous.

   **Proof:** (i) Given $V \subseteq Z$ open in $Z$,
   $V = V' \cap Z$ for some open set $V' \subseteq Y$.
   Then note that $g^{-1}(V) = f^{-1}(V')$, which is an open subset of $X$.

   (ii) Left as an exercise.

4. **"Local" formulation:** The map $f: X \longrightarrow Y$ is continuous if $\exists$ a collection $\{U_\alpha\}_{\alpha \in \Lambda}$ of open sets in $X$ s.t. $\bigcup_{\alpha \in \Lambda} U_\alpha = X$ and $f|_{U_\alpha}$ is continuous for each $\alpha$.

   **Proof:** Assuming such a collection exists,
   $\forall V \subseteq Y$ open,
   $$f^{-1}(V) = \bigcup_{\alpha \in \Lambda} \left(f^{-1}(V) \cap U_\alpha\right) = \bigcup_{\alpha \in \Lambda} (f|_{U_\alpha})^{-1}(V)$$
   Note: each set $(f|_{U_\alpha})^{-1}(V)$ is open in $U_\alpha$ $\because f|_{U_\alpha}$ is cts. Since $U_\alpha$ is open in $X$, each of these sets is also open in $X$, and thus their union, $f^{-1}(V)$, is open in $X$.

5. **Composition:** Let $f: X \longrightarrow Y$ and $g: Y \longrightarrow Z$ be continuous. Then $g \circ f: X \longrightarrow Z$ is continuous.

   **Proof:** $\forall V \subseteq Z$ open,
   $$(g \circ f)^{-1}(V) = f^{-1}(g^{-1}(V)). \because g^{-1}(V) \subseteq Y \text{ is open},$$
   $f^{-1}(g^{-1}(V)) \subseteq X$ is open.

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## Homeomorphisms

**Def:** Let $f: X \longrightarrow Y$ be a bijection. The map $f$ is said to be a "homeomorphism" if $f$ and its inverse $f^{-1}: Y \longrightarrow X$ are both continuous.

**Eg:**
1. **Non-constant linear functions on $\mathbb{R}$:**
   Consider any function $f: \mathbb{R} \longrightarrow \mathbb{R}$ given by $f(x) = mx + b$ where $m \neq 0$.
   Then $f$ is a homeomorphism.

2. Consider the function
   $$f: (-1, 1) \longrightarrow \mathbb{R}$$
   $$x \longmapsto \frac{x}{1 - x^2}$$
   This is a homeomorphism.

   **Proof:**

3. Let $B_n = \{x \in \mathbb{R}^n : d(x, 0) < 1\}$.
   Then the function $F: B_n \longrightarrow \mathbb{R}^n$
   $$x \longmapsto \frac{x}{1 - (d(x,0))^2}$$
   is a homeomorphism.

   **Proof:**

**Exercise:** Show that any two open intervals in $\mathbb{R}$ are homeomorphic.

**Non-example:** 1. Consider the identity function
$$f: \mathbb{R} \longrightarrow \mathbb{R}_\ell. \quad [f(x) = x \ \forall x \in \mathbb{R}]$$
Then $f$ is not continuous: $[a, b)$ is open in $\mathbb{R}_\ell$, but $f^{-1}([a,b)) = [a, b)$ is not open in $\mathbb{R}$.

**Exercise:** For this $f$, show that $f^{-1}: \mathbb{R}_\ell \longrightarrow \mathbb{R}$ is continuous.

2. Let $S^1 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ denote the unit circle. Consider $f: [0, 1) \longrightarrow S^1$ given by
   $$f(t) = (\cos(2\pi t), \sin(2\pi t)).$$
   Then $f$ is continuous and bijective, but not a homeomorphism.

   **Proof:**

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## Imbeddings

**Def:** Let $f: X \longrightarrow Y$ be an injective continuous map. Suppose $Z = f(X)$ and consider $f': X \longrightarrow Z$ obtained by restricting the range of $f$.
Note: $f'$ is a bijection.
If $f'$ is a homeomorphism, we say $f$ is a "(topological) imbedding" of $X$ in $Y$.

**Eg:**
1. Let $X, Y$ be topological spaces, and suppose $y_0 \in Y$.
   The map $f: X \longrightarrow X \times Y$ given by $f(x) = (x, y_0)$ is a topological imbedding.

   **Proof:**

2. Let $X$ be a topological space and $A$ be a subspace of $X$. The inclusion $i_A: A \hookrightarrow X$ is an imbedding.

   **Proof:** Exercise.

**Non-eg:** 1. Consider $f: [0, 1) \longrightarrow S^1$
$$x \longmapsto (\cos(2\pi x), \sin(2\pi x))$$
as above. The function $g: [0, 1) \longrightarrow \mathbb{R}^2$ obtained by expanding the range of $f$ is not an imbedding of $[0, 1)$ in $\mathbb{R}^2$.

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## Coordinate Functions

**Theorem:** Let $f: A \longrightarrow X \times Y$ be of the form $f(a) = (f_1(a), f_2(a))$ for some functions $f_1: A \longrightarrow X$, $f_2: A \longrightarrow Y$.
[$f_1, f_2$ are called the coordinate functions of $f$.]
Then $f$ is continuous $\iff$ $f_1$ and $f_2$ are continuous.

**Proof:**

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## Uniform Limit Theorem